Exercise 54

Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f, f', and f'' on a common screen and check to see if your answers are reasonable.

$$f(x) = x^3 - 3x$$

Solution

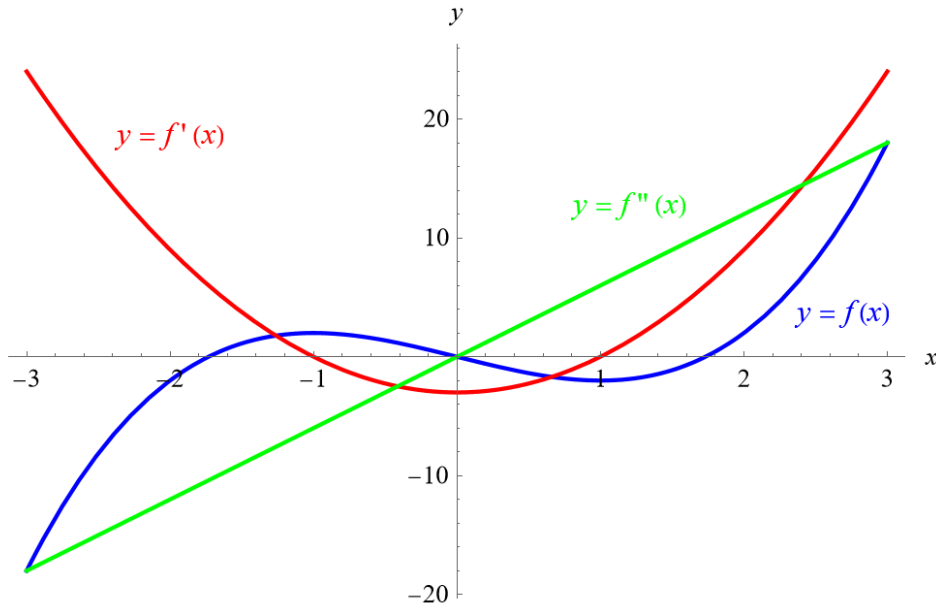
Use the definition of the derivative to find f'.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h)] - [x^3 - 3x]}{h} \\
&= \lim_{h\to 0} \frac{[(x^3 + 3x^2h + 3xh^2 + h^3) - 3x - 3h] - x^3 + 3x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 3) \\
&= 3x^2 - 3
\end{aligned}$$

Use the definition of the derivative again to find f''.

$$\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h\to 0} \frac{[3(x+h)^2 - 3] - [3x^2 - 3]}{h} \\
&= \lim_{h\to 0} \frac{[3(x^2 + 2xh + h^2) - 3] - 3x^2 + 3}{h} \\
&= \lim_{h\to 0} \frac{(3x^2 + 6xh + 3h^2 - 3) - 3x^2 + 3}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2}{h} \\
&= \lim_{h\to 0} (6x + 3h) \\
&= 6x
\end{aligned}$$

Below is a graph of $f(x)$, $f'(x)$, and $f''(x)$ versus x.



Notice that the red curve is negative where the blue curve is decreasing and positive where the blue curve is increasing. Also, the green curve is negative where the red curve is decreasing and positive where the red curve is increasing.